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THE VAPOR PRESSURE OF LIQUID WATER THAT HAS RECENTLY BEEN FROZEN

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In a paper appearing in this journal earlier in the year,[1] and which he referred to in a letter to *Nature*[2] of May 7, T. C. Barnes reported that the behavior of the alga Spirogyra in "water rich in trihydrol obtained from melting ice" was different from the behavior of the same alga "maintained under the same external conditions in water from the same source containing a lower proportion of polymerized molecules." If it were the case that the constitution of liquid water which had recently been frozen differed from that of other water, then one would expect that this difference would be shown in some of its physical properties. For this reason the experiments outlined below were undertaken to study a property of water reputedly sensitive to such postulated constitutional differences, namely the vapor pressure.

It might be supposed *a priori* that any appreciable difference of vapor pressure of different samples of water could not have escaped notice in the past. But it must be recalled that, so recently as 1927, H. B. Baker,[3] in his presidential address at London to the Chemical Society, reported precisely such differences of vapor pressure in the case of acetic acid, bromine and other pure liquids which he attributed to certain samples having previously been maintained at elevated temperatures while others had not. Had this actually been the case for pure liquids, then all existing measurements of vapor pressure of such liquids would have been in need of revision in the light of such an important finding. That Baker had been misled, however, was shown by West and Menzies[4] and by Wright and Menzies.[5] In a similar way, two other unorthodox beliefs dependent upon refined measurement, namely, that the boiling points of liquids and the dissociation pressures of certain solids are found with different values dependent upon whether the method used be static or dynamic, had many champions and much longer lives. Although both these anomalies, small in amount but apparently genuine, were likewise later disposed of by Menzies,[6] it is always

well to keep an open mind in regard to even the simplest phenomena; and the observations of Barnes led us to devise a differential pressure apparatus to test the case of water under conditions that should be similar to his.

We avoided entirely the use of stopcocks and of mercury by using a very simple apparatus made by connecting two 25 cc. Pyrex glass bulbs by a U-tube of capillary bore. Water itself, partially filling this U-tube, served as manometric liquid. Connection between the two bulbs could be cut off gastight by freezing the water in this manometer, easily effected by immersing this portion of the apparatus in a bath cooled by solid carbon dioxide. The apparatus was freed from permanent gases and charged with distilled water, about 10 cc. in each bulb, by a process of repeated boiling out through a steam exit tube itself dipping under boiling water in another vessel. After sealing off this exit tube, the apparatus was immersed in a thermostat. No difference in pressure between the two bulbs was observed. The error of observation, made in terms of water and converted to mercury by computation, was less than 0.05 mm. of mercury. To and fro distillations of the water did not alter this result.

The manometer liquid and the water in one bulb were now frozen solid. The ice in the bulb was then allowed to melt again, and the resulting water permitted to reach the bath temperature, after which the manometric water was permitted to melt. The pressures in the bulbs were now again compared. Temperature equilibrium between the two bulbs required, with shaking, fully five minutes within the thermostat from the moment of the melting of the last of the ice.

In one experiment the thermostat bath was maintained at 25°. No pressure difference was observable between the two bulbs. In the hope of favoring the persistence of polymerized molecules, the bath in another experiment was maintained at 3.5°. Again no pressure difference was observable. In each case the "comparison" water in the non-frozen bulb had been recently boiled.

For distilled water in glass apparatus, one must conclude from these experiments that, reaching equilibrium temperatures whether, of 25° or of 3.5° by direct approach, water which was solid ten minutes earlier exhibits a vapor pressure not different by more than 0.05 mm. mercury from the vapor pressure shown by water cooled directly from 100°.

[1] Barnes, T. C., these PROCEEDINGS, **18,** 136 (1932).
[2] Barnes, H. T., and Barnes, T. C., *Nature*, **129,** 691 (1932).
[3] Baker, H. B., *J. Chem. Soc.*, **951** (1927).
[4] West, W. A., and Menzies, A. W. C., *J. Phys. Chem.*, **33,** 1893 (1929).
[5] Wright, S. L., and Menzies, A. W. C., *J. Am. Chem. Soc.*, **52,** 4699 (1930).
[6] Menzies, A. W. C., *J. Am. Chem. Soc.*, **32,** 1623 (1910); and *Ibid.*, **42,** 1951 (1920).

THE ORIGIN OF THE LANDLOCKED HABIT IN SALMON[1]

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The salmon is recognized as the most valuable and highly prized of all fish. In nature its distribution is confined to the northern hemisphere north of 40° N. L. Only one species (*Salmo salar*) is found in the northern Atlantic. To spawn it ascends rivers both on the European and on the American coasts. Five species belonging to the genus Oncorhynchus are found on the Pacific coast of North America, and at least one additional species occurs on the northern Asiatic coast of the Pacific. All these fish have the similar habit of passing the major part of their life in salt water and ascending fresh water streams to find spawning grounds in the upper reaches. However, in some of the river systems that carry these salmon, certain fresh water lakes are known in diverse regions bordering the northern Atlantic and Pacific oceans, which contain salmon that never run to salt water. These fish, sometimes recognized by systematists as separate species, are generally regarded as descendants of the seagoing types which they resemble very closely. They are much smaller than their sea-run relatives and are sometimes spoken of as dwarf salmon. They are generally designated as landlocked salmon. The lakes in which they are found are widely scattered and unrelated to each other. Such landlocked salmon occur usually in elevated lakes and belong to both the Atlantic and the Pacific genera.

Landlocked Forms of the Genus Salmo.—The forms which belong to the genus Salmo may be considered first. The most extended study of a landlocked salmon yet made has been published[2] by Dahl (1927). The form in question is found in an elevated lake in southern Norway, 84 km. from the ocean. Dahl states that access to the lake is cut off by a waterfall only 15 km. from the sea. The original study made in 1914 led him to believe that the specimens were the result of fortuitous planting of salmon fry or else a landlocked form of *Salmo salar*. Ten years later other specimens taken from the same water possessed exactly the same characters as those previously examined and he concluded that they were landlocked. This type is common in the watershed of the Otra, occurring at several places other than that at which it was first captured. It is the object of seasonal fishing which is celebrated in a local rhyme of very ancient origin indicating thus its age. Dahl records in detail the specific characters of this fish, its habits, food and other biological features. A study of the stream showed that barriers now existing in the river prevent access from the sea to the region in which these salmon occur, and Dahl was led to the conclusion that

according to current ideas of the geological chronology of this region, the immigration and landlocking of this form occurred, roughly speaking about 9000 years ago. In personal correspondence Professor Dahl writes that he has found this landlocked or dwarf type in the waters of another river system in that country and believes it occurs in a few other rivers in southern Norway where elevated lakes of a similar character are to be found.

Corresponding dwarfed, non-seagoing or landlocked forms of *Salmo salar* have long been known in North America. Lake Sebago, Maine and some other waters in New England are inhabited by a type known as the Sebago salmon so similar to the Atlantic species that it is difficult to separate the two forms.[4] However, the Sebago type never descends to the sea although the outlet to Sebago Lake is not barred by any impassable obstacles. This salmon is regarded as the same as the ouananiche, famous in literature as well as in sport and found in various waters in Quebec and Labrador. This species has been transplanted in the egg stage into Lake George and other eastern lakes as well as into glacial lakes of the Sawtooth range in Idaho and elsewhere in the mountain regions in the northwestern United States. It has become well established in certain places. In native waters it does not go to sea though in some lakes it spawns in the outlet. When introduced into new waters this landlocked salmon sometimes shows a tendency to desert them and then it does not return. This may indicate that factors which inhibit movement seaward are absent in the new environment.

Salmo Salar in the Southern Hemisphere.—In nature all salmon are limited to the northern hemisphere and numerous attempts have been made to transplant both genera to southern waters. According to C. Tate Regan[6] the Atlantic salmon was introduced in 1864 into Tasmania but the experiment was not successful as the ocean waters were probably too warm. The experiments in New Zealand resulted very differently. A fine run was established in Lake Te Anau on the Waiau River near the southern end of the south island. Regan to whom specimens were sent states that they are certainly *Salmo salar*, resembling somewhat the ouananiche, and adds: "There appears as yet to be no evidence that the salmon of the Waiau system ever goes to sea." His view was confirmed by Hutton's study of the scales. The lake is some thirty miles long, very deep and has an abundant supply of small fish for food.[7] This is suggested as the reason why the salmon stop there and do not go down to the ocean. These fish are clearly landlocked salmon even the unintentionally so. Some minor peculiarities of structure and habit described are no doubt related to the food supply and the character of the waters they inhabit; so one may well regard it as a new species in the making.

Little is known of the conditions existing in the lake and of the habits of

the salmon in this new environment; hence judgment must be suspended on the validity of the explanation suggested for the origin of the landlocked condition. While the abundance of small native fish in the lake certainly serves to maintain the plant of salmon, it is not safe to conclude that these fish fail to continue their migration to the sea because of the abundant food supply in the lake.

The steelhead, often spoken of as a trout, belongs to the genus Salmo. While a migratory species that often inhabits seasonally both fresh and salt water, it accommodates itself readily to a permanent fresh water home and the landlocked type is found in many places in the Pacific coast region. A subspecies, the Kamloops trout,[3] "has been landlocked for so many generations that it has no inclination to return to the sea." (Keil 1929.) I have found no records of exact observations on the habitat of this species or of experiments with eggs or fry of the Kamloops trout.

The Pacific Salmon and Its Habitat.—The outstanding difference in habit between the Atlantic salmon, or the genus Salmo, and the Pacific genus Oncorhynchus lies in the fact that, whereas the former after spawning returns to the sea to feed there, and at a later date repeat the migration, the Pacific salmon spawns but once. All the fish both male and female make but a single trip to the spawning grounds and all die at the end of the reproductive period. The race is replenished from the progeny that spring from the eggs deposited at this single reproductive period. No adults ever go down-stream to the ocean.

The Pacific salmon seek spawning grounds in the upper reaches of rivers which come from glacier fed or spring fed lakes and streams in the high mountains. In their natural state these rivers are turbulent, rapidly flowing streams. Even though in places stretches of quiet water occur the current is still clearly in evidence. Under the conditions indicated the water is richly supplied with oxygen and is kept at a constant temperature. The temperature of the stream rises gradually as the water moves down to the sea, although under certain conditions occasional diurnal fluctuations may be rather marked. Thus in a canyon where the afternoon sun is reflected from a rocky face full into the stream the water temperature on an extremely hot day may be strikingly high in the midafternoon but as the sun sinks toward the horizon usual temperatures are rapidly restored. Because of their rapid descent from higher levels these rivers offer marked advantages for the development of water power projects and some sites on important salmon streams have already been provided with dams while many more dams are projected.

In studying groups of animals under natural conditions it is difficult to employ the experimental method because of the magnitude of the factors involved. In nature conditions are modified but usually too slowly to permit of ready observation of the effects of change. Thus it is impossible

ordinarily to modify stream conditions experimentally. However, in some cases industrial developments bring about radical changes on a large scale as has in fact been effected on certain salmon streams. These changes follow the erection of power dams. The relation of these changes to the salmon is most intimate.

The extremely radical character of the changes brought about by the erection of a power dam needs to be reviewed in some detail. Viewed in the large the stream bed may be considered as a plane descending from the lake where the sockeyes were spawned to the sea. The average depth, velocity, oxygenation and temperature of the water moving down this plane are subject to minor variations, but do not vary greatly. In other words, the environment which influences the young sockeye is relatively stable.

Into the plane of water moving rapidly and under stable conditions the power dam by forming a reservoir introduces a huge wedge of nearly static, i.e., not flowing, water presenting very different and previously entirely unknown environmental conditions. The inflowing water is small in quantity compared to the volume of the reservoir and the outflow is also minimal. In consequence the stream current has been eliminated except for short distances near the inlet and outlet. In fact, the outlet current in a power plant reservoir often cannot be detected by any current meter except the latter be located very close to the intake of the power plant. Furthermore, this intake is situated at a considerable depth so that at the surface of the reservoir there is nowhere any current leaving the lake unless the water is being wasted at some gate or spillway. In contrast with the original stream in which the salmon lived, the reservoir is a large mass of deep, quiet water.

Temperature conditions are also greatly modified. The summer heat warms up the surface water in the reservoir to a level probably never reached previously even under extreme and brief conditions in the stream, and this high level is maintained at the surface for a long period. Furthermore, the winter temperatures which had disappeared in surface water at the season when the young salmon deserted the original lake home are maintained in the depths of the reservoir since in midsummer the deep water is at or near 4°C. The stream water entering the reservoir at its head does not flow along the bottom but sinks only as far as its own temperature level and the water below that remains unchanged until the stream temperature drops in the fall. The water of the reservoir is warmer at the surface and colder in the depths during the migratory season than young salmon had found in the river before the construction of the dam.

Of the five North American species of the Pacific salmon only one is well known to occur as a landlocked type.[5] This is the little red fish or silver trout (*Oncorhynchus kennerlyi*). It is found naturally in numerous

lakes in the mountains from Alaska south to Oregon and is reported also from Hokkaido, Japan. The parent species spawns regularly in a lake and that habit as will be shown would contribute to its frequent occurrence as a landlocked type. Fish culturists have planted the latter widely as silver trout. The origin of this species is generally recognized by ichthyologists and fish culturists, many of whom refer to it specifically as the fresh water descendant of the big red fish, also called the blue-back salmon or sockeye (*O. nerka*). It lives and propagates well in many fresh water lakes and is thoroughly acclimatized to a fresh water existence. In details of structure except as to size, and in habits, the landlocked form agrees with the sea-going sockeye.

When the migrating young sockeye is studied under unmodified natural conditions one observes a definite sequence of reactions. After living in the lake in which they are hatched or planted for about a year, they move toward the outlet and pass into the outflowing stream. Once embarked on the down-stream journey they continue moving with the current leisurely until they reach salt water. Here they disappear into deep water which constitutes the habitat of the sockeye during the active growing period of the life cycle. From these feeding grounds they migrate two years later upward into surface waters and shoreward into fresh water, where the reproductive function is discharged. The new conditions created by the erection of a dam modify conspicuously these habits. I have had an opportunity to study such a case and wish next to record the recent results of that study. Two previous papers[9,10] (Ward, 1927, 1929) have given data secured earlier and only special or more recent observations bearing on the problem under discussion are included here.

Studies on Baker River Sockeyes.—The sockeye salmon spawns in very few places in the state of Washington. The most valuable run there ascends the Skagit and Baker rivers to spawn in Baker Lake. I studied these fish first in 1912 when natural stream conditions were mostly unmodified and have followed their relations to the changes in the Baker River associated with the modifications in their surroundings; these are particularly those due to the erection of a high power dam at Concrete, Washington. As the result of these studies, I ventured to predict in 1927 that the life cycle of the fish would be radically effected by this sudden change in stream conditions and that probably one of the results would be the production of a landlocked variety of the sockeye. A year later a good many such fish were caught on the sockeye spawning grounds at Baker Lake. The number increased in the following year and fell off considerably in 1930. In that fall only 10 such fish were taken at Baker Lake and all were males. In 1927 some and in 1928 and 1929 considerable numbers of such fish were seen spawning along the lake shore where the sea-run sockeyes were wont to spawn before the hatchery was established. In 1931 only 18 were taken.

all males; none were observed spawning. This evidence though too scanty to be conclusive suggests that the landlocked type consists preponderatingly of males. It indicates also that a regular run of these fish cannot be depended on. Perhaps conditions at the dam in 1928 were favorable for the escape of all the migrators and none became landlocked. However, the number of landlocked sockeyes in Lake Shannon is larger than indicated by the figures just given. Local fishermen recognize this type among fish caught in the lake at certain seasons. Some such fish have been taken with eggs in early stages of growth and at points which might indicate that they were on the way to spawning grounds in Baker Lake. The evidence is too incomplete to justify any conclusions as to the frequency of the landlocked sockeye in Lake Shannon or to determine whether all go to the ancestral spawning areas in Baker Lake.

The fish were clearly a dwarf variety of the sockeye and corresponded closely to the landlocked variety of that species as found in other waters. Eggs from these dwarf sockeyes have been secured, fertilized and hatched in different years. They have produced typical sockeye fry. In 1929 a total of 8615 fingerlings were raised from such eggs.

An analysis of the physical features of the region shows some basis for this fluctuation in numbers. The usual variations in climatic conditions have manifest themselves even in the few years that have elapsed since the dam was built. The total annual precipitation, and also its distribution by seasons especially, determine the water level in the reservoir. If this is high and water is wasted at the time when the migrators reach the dam, these young fish pass on down-stream with little delay. If, on the contrary, the reservoir is low and water is infrequently or not at all discharged, then the migrators are held at the surface of the lake. Under such circumstances some young fish descend into deep water in the lake. They remain there during the growth period of the life cycle until they attain the adult condition and go up to the spawning grounds. At this time they are evidently sockeye salmon, dwarfed and landlocked by virtue of the conditions that exist in the lake.

These adult landlocked fish varied from 6 to 12 inches, averaging about 10 inches in length. When first caught their blue backs, marked with small black spots and the silvery bellies gave the fish a likeness to the sockeye that was instantly noticed. At spawning time the males were very thin; most of them were red on back and sides with the black spots still showing. The females had only a little coloring and that of a darker shade. The females were found to be ripe and spawned in October and November. The eggs hatched between March 8 and May 24. The study of the scales showed that the ripe males were 3 years old and the spawning females were 4-year fish. They constitute unquestionably a new race of sockeyes passing their entire life in fresh water and properly designated as landlocked. The

modified conditions which produced this race, experimentally as it were, serve to explain the origin of the landlocked races found in nature.

Origin of Races Naturally Landlocked.—Factors that interfered with the seaward movement of the young sockeye might be either (1) such as prevented the fish from leaving the lake, or (2) such as enticed it to remain. Evidently these influences might be equally well physical or physiological. Only two factors have been previously suggested; of these stream barricades, especially falls, come under the first category noted above and a rich food supply under the second. My study of the cases known leads to the conclusion that neither of these is adequate. I purpose to show that the ruling factor is temperature.[8] Each of the factors may be discussed in detail to bring out values in each.

When one comes to consider the factor which has conditioned the change from the dual existence to a permanent fresh water habitat one finds difficulty in accepting the mechanical explanation suggested in the expression landlocked. As a matter of fact there is in many cases no impassable barrier between the fresh water lake in which these fish live and the salt water. At Sebago Lake the entrance to the river is free and open, the distance to the ocean relatively short and despite a small dam found in the stream today, the journey would be easy for any salmon that essayed to make the trip. In the lakes of the Pacific coast the distance though greater is not more than covered by other salmon in other streams of the same region. In some streams no obstacles of a serious nature oppose either the descent of the migrators or the ascent of the adult fish so that any mechanical interpretation of the situation seems untenable.

Even in those cases in which the stream is barred by impassable falls or rapids so that no salmon coming from the sea could possibly gain access to the waters in which a landlocked variety is found today, the origin of a landlocked variety can hardly be traced to the obstacles in question. If these barriers were formed gradually as is likely in most cases, the upstream journey would be barred long before it was even dangerous to go down-stream. More than that if the young salmon had not lost its urge to go down to the sea or been held back by a stronger impulse, it would have followed with the current even had that carried it over one of the impassable falls which we are told block the way at present. When one sees young salmon, moving seaward by thousands, go over the spillway of a high dam without hesitating, it is evident that no sense of impending danger retards their movements. One must conclude that the salmon which tarry in those elevated lakes that are the home of the landlocked types remain there in response to present-day influences that hold them back. If they are descended from forms which once went down-stream to the ocean, some later generations were impelled at some time to give up the urge to go with the current and desert the lake by some more powerful influence, some at-

traction that held them back or turned them aside and thus kept them from carrying out their primal impulse. If there had not been some such influence all the salmon would have gone on out of the lake and down to the sea just as have the generations of young salmon in other streams in the same region. The landlocked type resembles sea-run species so closely that at first sight men recognized their relationship; the name given to these fish represented the simplest explanation of their origin, but as suggested it seems to be inadequate to account for present conditions. If ancestral habits had ruled, the lake would be barren today of these fish.

These thoughts led me to search for a more adequate cause. Another and better interpretation of the situation did not offer itself at once. I recall distinctly that while on Lake Sebago, Maine, many years ago, I talked of this matter with Dr. W. C. Kendall who was also there studying the Sebago salmon. This type was so like the Atlantic salmon (*Salmo salar*) that it was difficult to find criteria to separate them, yet the Sebago salmon never went to sea even though the way was wide open. We could not see why and had to satisfy ourselves with the comment that probably the type was physiologically landlocked. Only recently has evidence come to me which seems to provide an adequate explanation of the phenomenon. That evidence was secured while studying the behavior of young sockeye salmon under changed conditions in the Baker River following the erection of a power dam at Concrete.

These conditions have already been described in good part. When the migrators, dropping down-stream from Baker Lake reach the reservoir, Lake Shannon, they move along the shore seeking an outlet. Under natural conditions they should find one and make their way out but when the water is low and no discharge from the surface exists, the migrators linger there until with advancing summer conditions, the surface water becomes warmer than the species likes. It has already been pointed out that in the reservoir the surface water heated by the summer sun reaches a temperature considerably warmer than the flowing stream ever attained. There is only one way for the fish to escape the warm waters and they take it. They move downward to cooler levels and disappear from the surface zone. How far they go could not be ascertained but probably they move below the thermocline which forms soon after the time at which they disappear from the surface. With that they have lost all chance of leaving the lake and continuing the journey to the sea. Nothing further is known of them until sexually mature they appear at the spawning grounds of the species. They have lived and grown to maturity in the reservoir and like their sea-run ancestors have responded to approaching maturity by moving up-stream against the current to the place where fulfilment of the reproductive function terminates their existence. In this manner there has been produced experimentally a race of landlocked sockeyes.

Origin of a Natural Race of Landlocked Salmon.—The question now arises —how do conditions in nature resemble those which were artificially created by the erection of the dam, and do somewhat similar influences ever interfere to prevent the young salmon from completing the journey to the sea? Earlier in this paper attention was called to the fact that landlocked salmon planted in certain lakes tend to desert those waters. One will naturally assume that in such cases some influence is lacking which in other localities inhibits the fish from deserting the lake. I am of the opinion that temperature is the ruling factor and that changes known to have occurred in the recent geological history of the northern hemisphere are calculated to afford an adequate explanation of the formation of landlocked races of salmon in widely scattered lakes.

During the period of glaciation the salmon, attracted no doubt by the fresh water inflow from ice-fed streams, must of necessity have spawned nearer the sea. In the changes that were wrought as the glaciers retreated some lakes near the shore were cut off from connection with streams and being spring fed kept their outflow, ascending which came the migrating sockeye to spawn in the lake. In other cases the salmon followed the retreating glaciers into newer and higher reaches of the streams and the sockeye found there often lakes for spawning but did not reach *all* lakes, for many with suitable spawning areas have never sheltered a run of red salmon even though the species occurs abundantly in other parts of the same river system. Conditions in these streams should be studied to ascertain whether unfavorable temperature conditions do not in general bar access to such lakes as they apparently do in isolated instances which I have found. But that is another problem and need not be discussed here. It is necessary to follow further the life cycle of the sockeye as climatic changes occurred which affected this lake spawning species.

The adult sockeyes spawned in these cold lakes and the young salmon when hatched found in the lakes sufficient juvenile food to hold them over the first year. In the streams where other species of Oncorhynchus spawned the food supply was of course very scanty and so the young seeking food drifted down-stream earlier and went out to sea in their first year. The lake feeding young sockeyes stayed in that environment until the following year when in the spring the urge for more food—or different food —led them to leave the lake and journey down-stream toward the ocean. So long as the water remained cold in the lake when they were spawned or in lakes lower down on the stream through which they had to pass, the journey to the sea was uninterrupted. But with the gradual melting of the ice the climate warmed up and changes occurred in the water. While the streams became warm, the temperature was uniform in the entire mass and they were still glacier fed so that the water remained relatively cool.

One can speak of temperature changes in exact figures only at the present.

When the surface water of the lake passes 10°C. the migrators which have been continuously at or near the surface appear to desert that level and withdraw into deeper layers. Much before the surface temperature has reached 15°C. the fish are no longer seen; apparently they have taken up their abode permanently in deeper, cooler waters. The conditions seen today in these lakes are the result of gradual changes in climate following the glacial period. As the summer became longer and warmer, the temperature of the surface water rose and the critical temperature for migrating salmon was reached earlier in the season. When conditions were reached where the salmon were influenced to seek deeper, cooler waters before they had reached the outlet of the lake, or when they found the outflowing stream unattractive because its temperature was to them too high, then they abandoned the movement down-stream and seeking deeper, cooler waters were at the end of their migration.

In the lakes conditions were different. With the gradual climatic change the lake water also became warmer in summer than it had been previously. But in the absence of a current adequate to mix the mass thoroughly the surface water reached a higher temperature than was found in the stream. The effect of this on the movements of the sockeye depended on the relation between the time of the salmon migration and the amount of change in temperature of the water.

The exact time at which the temperature changes took place varied then as now with the lake and the year. If the young sockeyes passed out of the lake, or through it as the case might be, before the temperature had risen unduly the migratory movement was not interfered with. But if the location of the lake, its exposure to the sun, its scanty outflow or any other factor, warmed up the surface layer before the sockeyes deserted the lake, something else happened.

It may be that in New Zealand the salmon were held in Lake Te Anau by a superabundant food supply as suggested, but similar conditions do not exist in lakes in North America which contain landlocked salmon. In fact reports from certain of these lakes indicate a scanty food supply and especially an entire lack of small fish which were conspicuously abundant in the New Zealand lake. Furthermore, the migrators, or year-old sockeyes, leave Baker Lake when the food supply is still there and the young fish recently hatched are appearing in large numbers, thus making a new and richer food supply. The exact amount of plankton food has not be measured in any of these lakes so far as I can ascertain, nor is its seasonal variation known. Yet what is known justifies us in rejecting the view that any abundance in the food supply is responsible for the landlocked habit.

Conclusions.—In the light of all the evidence one may conclude: The landlocked salmon in a lake were not influenced or held in restraint by the formation of any mechanical barrier that cut off access from the ocean

but by some factors within the lake itself which changed primitive conditions. Some change brought about the disappearance of the current stimulus that would have led them on down-stream to the ocean. These new conditions caused them to hesitate and finally to abandon their journey to the sea. In the Baker River it was the erection of the dam which created a great body of quiet water and thus eliminated the current stimulus. Following that it was the warming up of surface waters which led the young fish to seek deep, cool waters. One should recall also that when in the old unmodified river the migrators had reached the sea, the next move of the young fish was to turn there into deeper, cool waters. The succession of responses is the same in both cases. The changes brought about by the erection of the dam were rapid and the results immediate.

Somewhat similar modifications have resulted from climatic changes in geological eras recognized and recorded in the history of the regions in which landlocked salmon are now found in nature. Reduction in stream flow in many places has left a geological record that is plain. As the volume of water in the stream has decreased gradually, the outlet has been silted up or been blocked by land slides. Seasonal variations in precipitation have also intervened to diminish the stream flow and reduce the current which guided the migrating salmon. As the region warmed up after the recession of the glaciers, the surface water in some lakes finally reached a temperature level unattractive to the cold water loving salmon. Then the fish sought cooler levels in the lake and were thereby prevented from continuing their journey down-stream.

Thus environmental changes in the course of time exercised ultimately similar influences on the salmon to those involved in the erection of the power dam. A large body of quiet water was intercalated in the course of a turbulent stream and the current stimulus disappeared; then the warm surface waters turned the salmon downward and once having left the surface they were held to carry out one part of the life history in the cooler depths of the fresh water lake instead of in the ocean. The migrating salmon had become physiologically landlocked.

[1] Contribution from the Zoölogical Laboratory of the University of Illinois, No. 430.

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THE IMPORTANCE OF THE TIME ELEMENT IN THE DETERMINATION OF THE RESPIRATORY METABOLISM OF FISHES

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In experiments involving the determination of the gas exchange of fishes it is necessary not only to handle them but to place them in an environment that is far from normal. As a result of this procedure, the respiratory metabolism is relatively high at first, but it gradually decreases with time. This initial high rate terminates usually, in from four to six hours. Most investigators (Gardner and Leetham, 1914; Ege and Krogh, 1916; Gaarder, 1918; Powers and Shipe, 1928; Hall, 1929, 1930; and others) have not taken into account this initial effect. Keys (1930) was the first to report it and he explained it as being due, in part at least, to the payment of an oxygen debt contracted by the fishes in their struggles incident to being placed in the respiratory apparatus, and possibly in part to sub-minimal stimuli (Winterstein and Hirschburger, 1927). He stated that "in order to be sure that the determinations represent standard or anything like basal metabolism, it is necessary to wait five or six hours after the fish has been placed in the apparatus before beginning the final determinations." He also assumed that if, after six hours in the respiratory chamber, two or more determinations over a minimum period of two hours checked reasonably well, they represented standard metabolism.

The purpose of the present paper is to attempt to show that, with *Fundulus parvipinnis*, even though there is a temporary equilibrium established five to six hours after the fishes have been installed in the respiratory chamber, this does not represent standard metabolism, but that after 24 hours or more a lower level is attained which is practically constant and represents "normal" metabolism.

In a long series of experiments carried out by the author to determine the effect of temperature on the rate of respiratory metabolism, it was found that this species, after a period of four to six hours in a respiratory chamber at a given temperature, usually strikes a metabolic level which it may maintain for several hours. It was many times noted, however, that the last determination indicated a lower level than any of the previous ones, which suggested that a normal level for the temperature in question had not been attained. In order to determine whether these frequent lower values were significant, a group of thirteen *Fundulus* which had been used a few days before were again placed in the respiratory chamber at the same temperature and kept there for one hundred hours. The

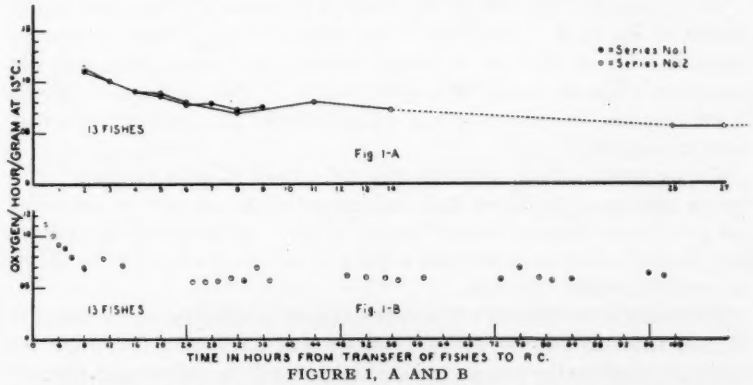

FIGURE 1, A AND B

The ordinates represent oxygen consumed in cc. per gram of fish per hour at 13°C.; the abscissas represent the number of hours from time of transfer of fishes to respiratory chamber. 1*a*—Series 1 and 2 plotted to the same scale. 1*b*—Series 2 with the time scale reduced to one-quarter, but ordinates the same as in 1*a*. The 13 fishes used in these two series were of approximately the same size and their weight averaged 6.5 grams.

values obtained from this and subsequent experiments will be set forth and discussed below. There were in all six series of oxygen determinations, each series representing a period of from 10 to 100 hours.

Material and Method.—The fishes used in the six series were *Fundulus parvipinnis*. The thirteen fishes mentioned above as having been used in the first of the long-time experiments (series 2) had been used in previous experiments and had been in the respiratory chamber seven days previously for a period of ten hours (series 1). Figure 1, *a* and *b*, contains the data for series 1 and 2. In 1*a* the data from the two series are plotted on the same time scale, in 1*b* the data from series 2 are plotted on a time scale one-fourth that of 1*a*. Groups 3, 4 and 5, the data for which are contained

in figure 2, were three different weight-groups averaging 1.8 grams, 3.82 grams and 6 grams, respectively. No individual deviated more than 0.2 gram from the average and in most cases less than that. Group 3′ (see Fig. 2) contains the same fishes as group 3, but an interval of thirty days had elapsed and their average weight had increased to 2.4 grams. The number of fishes employed in each series was determined by their size relative to the capacity of the respiratory chamber. As many as possible were used, in order to reduce the effect of individual differences. The capacity of the respiratory chamber allowed the fishes room enough in each case to move about to some extent and to become accustomed to their environment. In all cases, they were fed on a uniform diet and were transferred to the respiratory chamber at a definite time after being fed.

The apparatus used will be described in detail in a subsequent paper. Briefly, it provides a continuous flow, constant temperature system, in which the rate of flow can be maintained at any desired value, and the temperature can be controlled to within 0.1°C. After the fishes are placed in the chamber they are in no way disturbed until the series of determinations is completed.

The oxygen was determined by the well-known Winkler method. The rate of flow was so adjusted that the amount of oxygen used by the fishes was never more than a third of that available. Preliminary experiments have shown that if no more than a third of the oxygen is used the pH of the water is barely affected.

Discussion of the Results.—The first group of thirteen fishes, run August 10, and again on August 17, 1931, shows that after six hours in the respiratory chamber the oxygen metabolism in each experiment had reached approximately the same level, and the general trend of the two curves is nearly the same for the first nine hours, at which time the first series was discontinued. Figure 1 shows the successive values of the oxygen consumed. The ordinates represent the oxygen consumed per gram of fish per hour, and the abscissas represent the number of hours from the time of transfer of the fish to the respiratory chamber. Although the values for these two series are approximately the same, it will be noted that on the second day of the second series (Fig. 1*b*) the values had dropped considerably lower and maintained this lower level each successive day until the experiment was terminated. There is no indication of an increase or decrease in the metabolic rate. The temperature for these two series was constant at 13°C., but in the case of the first series the fishes had been kept at that temperature for three days before being transferred to the respiratory chamber. After removal from the chamber they were placed in running sea water in the laboratory, which at that time was about 23°C. In the second experiment with these same fishes, however, they were taken directly from water at a temperature of 23°C., transferred

to the respiratory chamber and the temperature immediately lowered to 13°C. I shall return to this point later.

The curves 3, 4 and 5 in figure 2 represent the data obtained from the three weight-groups as stated above. Curve 3 is based upon 29 small fishes, curve 4 upon 22 medium-sized fishes and curve 5 upon 19 full-grown ones. The curve 3′, figure 2, represents the results obtained from the 29 fishes of group 3 after an interval of 30 days. As the plotted data show, the respiratory metabolism of all four groups is considerably lower during the second twenty-four hours than during the first, and is likewise more constant. In three of the four cases the bi-hourly fluctuations which occurred during the first day were greatly reduced during the second

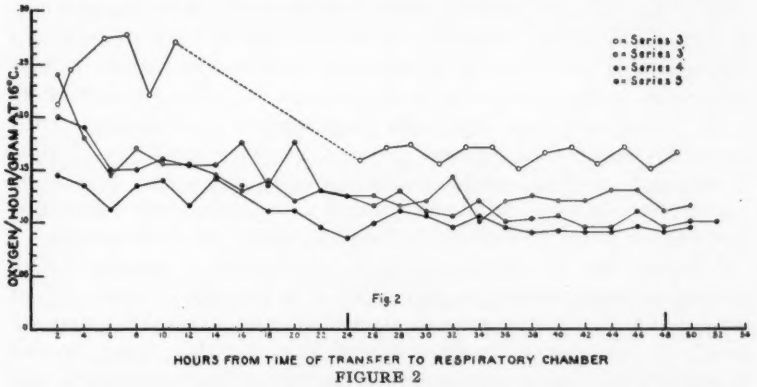

FIGURE 2

Ordinates represent oxygen consumed in cc. per gram of fish per hour at 16°C.; abscissas represent time in hours from transfer of fishes to respiratory chamber. Series 3 comprises the small fishes (weight 1.8 gm.); series 3′, the same fishes 30 days later (weight 2.4 gm.); series 4, the medium-sized fishes (weight 3.82 gm.); and series 5, the large ones (weight 6 gm.).

twenty-four hours. This is most strikingly illustrated by the small fishes (curve 3, Fig. 2). The temperature in all four cases was 16°C. This temperature was chosen because it was that of the running sea water in the laboratory at the beginning of the last four series of determinations.

In the first long series (series 2, Fig. 1), done in August, 1931, I made determinations of the oxygen metabolism during the day only. At night the fishes were in total darkness. As a glance at the graph (Fig. 1*b*) will show, the final drop to the "normal" level occurred sometime during the first night. In the second of the long series (series 3, Fig. 2), there was a period from 8.00 P. M. until 10.00 A. M. in which no determinations were taken. It was also during the night in this case that the respiratory metabolism dropped to the level that I shall call "normal" metabolism.

In the three following series, however (4, 5, 3′), determinations were made at intervals of two hours throughout the duration of the experiment. In doing this it was necessary to expose the fishes to continuous light, whereas in series 2 and 3 they were undergoing the normal sequence of light and darkness. In these three last series the fishes did not reach a "normal" level until twenty-four hours or more had elapsed. In the first two, on the other hand, the normal level was attained sometime during the night. If, as I feel justified in assuming, the attainment of a "normal" metabolic rate at a given temperature depends on the time required for the fishes to become accustomed to their environment, then the more nearly normal the environment the less time will be required to attain it.

I have called attention to the fact that the fishes used in the first two series (Fig. 1) had been subjected to different temperatures before being transferred to the respiratory chamber. In the first series they had been kept at a temperature of 13°C. for a period of seventy-two hours previous to the time of transfer; in the second series the temperature was lowered from 23°C. to 13°C. immediately after they were transferred to the chamber. Their respiratory metabolism, however, followed the same trend in each case. It was such excellent duplication of results as the above, even when starting from different conditions, that led Keys (1930) to the assumption that after six hours the fishes had attained a normal rate of metabolism.

Although this first level, apparently representing a temporary equilibrium, is quite evident in most cases, it is not always manifest. In series 4 (Fig. 2) the values for the second six-hour period are of approximately the same degree of magnitude, while in curve 3 there is no indication of such a level. In the second six-hour period the fishes in groups 3′ and 4 had approximately the same rate of metabolism, although those in group 3′ were much smaller. If these values had been taken as representing "normal" metabolism it is quite obvious that they would have led to an absurd conclusion.

I have defined "normal" metabolism of fishes as that value of the respiratory metabolism obtained at any given constant temperature within their normal temperature range and which is maintained for a period of at least 12 hours at approximately the same level. This, of course, presupposes the following conditions: (1) That the fishes are normal and healthy, (2) that the values for the respiratory metabolism are taken at a certain definite time after the fishes have been fed in order to insure a standard nutritional state, and (3) that the species, the weight of the fishes and the temperature shall be stated. Furthermore, I have found that with *Fundulus parvipinnis*, any sudden disturbance will cause an immediate increase in the rate of metabolism, the effects of which may last for an hour or more. Other investigators (Keys, 1930; Adkins, 1931) have emphasized the advantage of using the continuous flow system.

Although I feel justified in stating that fishes of this species do not reach a normal metabolic level until approximately twenty-four hours after they have been handled and transferred to a strange environment, I should not assume that this is true for other species of fishes. I should be doubtful, however, of the validity of data obtained on the respiratory metabolism of other species, unless the data covered a period of at least thirty-six hours, or unless the investigator had determined the time required for them to reach a "normal" level. For *Fundulus parvipinnis*, in view of the data for series 2, figure 1*b*, I feel confident that an average of values over a period of eight hours, after a lapse of twenty-four hours or more from the time of transfer to the respiratory chamber, represents "normal" metabolism. This is true at least for fishes weighing 6 grams, at a temperature of 13°C. Indeed, I should expect that with very active, "nervous" fishes several days would be required for them to reach a normal level of metabolism, while on the other hand, quiet, "sedentary" fishes may do so in a few hours.

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THERMAL DOPPLER EFFECT AND TURBULENCE IN STELLAR SPECTRA OF EARLY CLASS

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The broadening effect on spectral lines of the random motions of the atoms in an absorbing mass of gas has been thoroughly investigated theoretically, but has never been directly measured in stellar spectra. Following an important paper by W. Schütz,[1] it was shown by M. Minnaert and G. F. W. Mulders[2] that the integrated intensities of faint solar absorption lines fail to obey the classical expression of the absorption coefficient within the line, but that they are in harmony with Voigt's formula which takes account of radiation damping as well as of thermal Doppler effect. A similar use of the work of Schütz was made by A.

Unsöld, O. Struve and C. T. Elvey in their discussion of the interstellar calcium lines.[3]

These applications to astrophysical problems depend entirely upon the consideration of total absorbed energies: the actual line-contours are in all these cases so narrow that they cannot be regarded as free from instrumental effects; the optical properties of the instrument bring about a complete redistribution of the intensities, leaving only the total absorbed energy constant. This has been demonstrated by Unsöld[4] for the 75-foot spectrograph of the Mount Wilson Observatory. For ordinary prismatic stellar spectrographs, conditions are even less favorable. Thus, for the three-prism spectrograph of the Yerkes Observatory the resolving power is $R = 97{,}600$ and the purity of the spectrum is $P = pR$, where p depends upon the slit-width. For an infinitely narrow slit $p = 1$. Following the computations of H. F. Newall,[5] we find that for the focal ratio of the Yerkes instrument (1/19) and for a slit-width of 0.05 mm., $p \approx 0.15$. According to A. Schuster[6] two lines are completely resolved if

$$\delta\lambda = \frac{\lambda}{P} = \frac{\lambda}{pR}$$

and the separation begins to be noticeable at about one-half this value. Substituting our numerical results we find $\delta\lambda = 0.3$ Å at $\lambda = 4500$ Å. For single-prism dispersion $\delta\lambda = 0.9$ Å. We conclude, therefore, that a perfectly monochromatic line will have a width of

$$0.45 \text{ Å} < W < 0.9 \text{ Å (single prism)}$$
$$0.15 \text{ Å} < W < 0.3 \text{ Å (three prisms).}$$

Actual measurements of several comparison lines near λ 4481 give, in the mean, $W = 0.6$ Å for the single-prism arrangement. If the original line is not monochromatic, the resulting broadening will be superposed over that of the instrument, but it would be useless to search for effects which are much smaller than the instrumental broadening.

If the contour of an absorption line is defined entirely by thermal Doppler effect, the absorption coefficient is given by

$$\sigma = \text{const. } N.e^{-\left(\frac{\Delta\lambda}{\alpha}\right)^2} \qquad (1)$$

where N is the total number of atoms per cc., $\alpha = \frac{\lambda_0 v_0}{c}$, and v_0 is the most probable velocity corresponding to the given temperature T and atomic weight m:

$$v_0 = \sqrt{\frac{2kT}{m}}. \qquad (2)$$

The exponential factor in (1) shows that σ is proportional to the number

of atoms whose velocities in the line of sight are between $v\left(=\frac{\Delta\lambda.c}{\lambda_0}\right)$ and $v + dv$.

The contour of the line is, according to Unsöld

$$I/I_0 = \frac{1}{1 + \sigma H} \tag{3}$$

where H is the thickness of the atmosphere. It is simpler, however, to use the ordinary exponential formula

$$I/I_0 = e^{-\sigma H} \tag{4}$$

which differs from (3) in that it neglects the process of reëmission. In the wings σH is small, and by expanding (4) in a power-series and neglecting terms of $(\sigma H)^2$ and higher, we have

$$I/I_0 = 1 - \text{Const. } N.H.e^{-\left(\frac{\Delta\lambda}{\alpha}\right)^2}. \tag{5}$$

The absorption for any given $\Delta\lambda$ is therefore proportional to $N.H$, and we should expect to observe the broadening more readily for strong lines than for weak ones.

However, this result is not, in general, correct if broadening by radiation damping, or by collisional damping, is also present. The absorption coefficient which takes account of Doppler broadening as well as of radiation damping, has been given by Voigt.[7] An analysis of the resulting contours for the spectrum of the sun has been recently published by A. Pannekoek.[8] His results are, briefly, as follows:

1. For strong lines the wings are determined by the effect of radiation damping, and thermal Doppler effect produces merely a broadening of the inner core. For strong hydrogen lines ($m = 1$, $T = 6000°$, $\lambda_0 = 4000$ Å) the Doppler core has a half-width of about 0.5 Å; for calcium the corresponding value is about 0.09 Å.

2. For intermediate lines the wings produced by radiation damping contract in agreement with Unsöld's formula, but the width of the core remains almost constant.

3. For weak lines the effect of radiation damping is imperceptible. Wings and core approximate formulas (3) and (1). As the lines grow weaker their central absorptions decrease, and their wings contract.

4. Collisional damping produces an effect similar to that of radiation damping and tends to enhance the wings. This effect is probably negligible in the hotter stars.

5. Turbulence may be treated for any given element by assuming

$$v_0^2 = \left(\frac{2kT}{m} + \beta\right) \tag{6}$$

in place of (2). Therefore turbulence would increase the Doppler core by an amount which is independent of atomic weight.

Ignoring collisional broadening and turbulence, we may immediately apply Pannekoek's results to the spectra of stars of early type. The wings produced by damping are independent of T. On the other hand, the width of the Doppler core is roughly proportional to $v_0 = \sqrt{\frac{2kT}{m}}$. Thus for a B0 star $T \approx 24{,}000\ K$ and

$$\frac{v_0(H,\ T = 24{,}000)}{v(H,\ T = 6{,}000)} = 2.$$

The Doppler core should be approximately 2 Å wide.

In reality the hydrogen lines are greatly broadened by ionic Stark effect. It is probable, nevertheless, that the Doppler core is actually present. In a paper by Elvey and the author,[9] it was shown that the central portions of the hydrogen lines originate in levels where the pressure is extremely small, and where consequently Stark effect must be nearly absent. Only the wings originate in the denser layers, where the electric fields of neighboring ions are strong. As a result, all hydrogen lines, irrespective of spectral class, contain narrow cores superposed over broad and shallow wings. Tracings obtained with the microphotometer show that when no rotational broadening is present the contour comes to a sharp point in the center.

I have found from a visual examination of our best spectrograms that in stars of very early spectral class the core has approximately the width required by the theory of Doppler broadening. In stars of later spectral class the hydrogen cores seem to be narrower, as a rule, which is also in agreement with the theory. But this test is not fully satisfactory because it is not quite certain that Stark effect is to be entirely neglected in the cores.

The helium lines are better suited to this test: not only is there a large selection of strong and faint lines (which should enable us to select some in which the wings caused by radiation damping are absent and in which Doppler broadening is prominent), but many of them are practically immune to electrical fields. The theoretical width of the Doppler core for helium is approximately 1 Å (for $m = 4$, $T = 24{,}000\ K$). This is just below the effective resolution of our single-prism spectrograph, but should be observable with our three-prism dispersion.

The only line of helium, within the region covered by our three-prism spectrograph, which is suitable for the test is λ 4438. The laboratory results of Y. Ishida and G. Kamijima[10] show that for fields up to 100 kv. the Stark effect is hardly perceptible in this line. Two excellent three-prism spectrograms (on Eastman Process emulsion) of the B2 star γ Pegasi show that He 4438 is perhaps a trace more diffuse than are O II

4415, O II 4417 and Mg II 4481. An inspection of Plate XI, vol. 74 of the Astrophysical Journal tends to confirm this result: the helium lines λλ 4169, 4121, 3965, 3868 all look slightly wider than such lines as N II 3995, O II 4070 or C II 4267. The width of the core of He 3965, as measured on the original enlargement, is approximately 0.7 Å.

Two excellent single-prism spectrograms of τ Scorpii (B0) show that He 4121 is diffuse. However, it is possible that this is caused by blending with O II 4120.30 (int. 3) and with O II 4121.48 (int. 4) since ionized oxygen is strong in this star. The line He 4169 is not appreciably broadened on the single-prism plate.

From these two examples we may conclude that a linear dispersion of 10 Å/mm. at λ 4500 is barely sufficient to show the effect of thermal Doppler broadening in the hottest stars. A dispersion of about 5 Å/mm. should suffice to measure the effect in those helium lines which are immune to Stark effect.

The width of λ 3965 in γ Pegasi is, as we have seen, almost identical with that estimated on the basis of the theory of Doppler broadening ($\approx$ 1 Å). This shows conclusively that the turbulence factor β in (6) is insignificant.

A. Unsöld,[11] and more recently P. C. Keenan,[12] have shown that in the solar chromosphere the emission lines and their central reversals are broadened by turbulent motions. It is natural to ask whether turbulence becomes more prominent in the hotter stars, perhaps affecting even the deeper reversing layer. The great majority of early-type stars have slightly fuzzy lines, even when there is no measurable "rotational" broadening. The results for γ Pegasi and τ Scorpii show that in these two stars turbulence does not play an important part. Since both are perfectly normal for their spectral subdivisions we conclude that turbulence is not a phenomenon characteristic of early-type stars. It is, of course, possible that in individual stars turbulent motions are present, but it is distinctly more probable that the fuzzy lines in A, B and O stars are caused by rotation alone.

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ON THE INTENSITIES OF STELLAR ABSORPTION LINES

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1. The intensity of a stellar absorption line depends upon the number of absorbing atoms, the atomic absorption coefficient and the manner in which the light is propagated through the reversing layer. The ionization theory gives an expression for the number of atoms in the lower energy state of any given transition but does not directly give the intensity of the line. If the observations were perfectly represented by the equations of the ionization theory, we should be able to conclude that the other two factors are constant for all stars and for all spectral lines.

That this is not the case has already been shown for the helium lines in B-type stars;[1] the relative intensities of the triplet and the singlet lines of helium are not the same in all stars, although the excitation potentials are practically identical. We have now found that there are similar differences in the doublets and the quartets of singly-ionized oxygen. Furthermore, it appears that even in separate multiplets of *Fe* I, *Fe* II, *Ti* II, etc., the relative intensities of the lines are not the same in all stars. In some cases we have found striking differences in the intensity gradients within multiplets, and although we have at present no explanation to offer for the observed differences, it is clear that the problem of the intensities of stellar absorption lines is much more complicated than has been realized. This paper is a brief report of the preliminary observations. More detailed investigations are now in progress at the Yerkes Observatory, and will be published in the near future.

2. The spectrum of the star 17 Leporis [$\alpha\ 6^h\ 0^m.5\ \delta\ -16°\ 29'$ (1900), Harvard *AO;* photometric magnitude 5.0] has been compared with the spectra of ϵ Aurigae, α Cygni and several other stars. The lines of *Fe* II and *Ti* II are particularly strong in these stars and have been carefully investigated. The strongest absorption lines of both *Fe* II and *Ti* II are slightly stronger in 17 Leporis than in ϵ Aurigae; the weaker lines, on the contrary, are very much stronger in ϵ Aurigae. This is true for every multiplet which has strong lines. Multiplets in which even the principal members are relatively insignificant, are weaker throughout in 17 Leporis than in ϵ Aurigae. The effect is therefore not limited to multiplets: the intensity gradient of all lines of *Fe* II and *Ti* II is much greater in 17 Leporis than in ϵ Aurigae. Table 1 gives the intensities of the *Fe* II multiplet ($b^4F - z^4D^0$) as computed from the formulae of Russell[2] and as observed in 17 Leporis and ϵ Aurigae. In this table the sum of all intensities within the multiplet was arbitrarily made equal to 100. The

relative intensities of the lines in α Cygni resemble those of ϵ Aurigae and the same is true of nearly all other stars of similar spectral class. 17 Leporis is an exception; it has a number of other peculiarities which will be described elsewhere.[3]

TABLE 1

INTENSITIES IN *Fe* II MULTIPLET ($b^4F - z^4D^0$)

LINE	4583	4549	4522	4508	4620	4576	4541
Designation	$b^4F_{4\frac{1}{2}} - z^4D^0_{3\frac{1}{2}}$	$b^4F_{3\frac{1}{2}} - z^4D^0_{2\frac{1}{2}}$	$b^4F_{2\frac{1}{2}} - z^4D^0_{1\frac{1}{2}}$	$b^4F_{1\frac{1}{2}} - z^4D^0_{\frac{1}{2}}$	$b^4F_{3\frac{1}{2}} - z^4D^0_{3\frac{1}{2}}$	$b^4F_{2\frac{1}{2}} - z^4D^0_{2\frac{1}{2}}$	$b^4F_{1\frac{1}{2}} - z^4D^0_{1\frac{1}{2}}$
Comp. Int	36	25	16	10	4	5	4
$a\sqrt{\text{Comp. Int.}}$	25	20	16	13	8	9	8
ϵ Aurigae	22	19	15	13	8	12	11
17 Leporis	26	27	19	13	5	4	7
ϵ Aurigae/17 Leporis	0.8	0.7	0.8	1.0	1.6	3.0	1.6

3. Similar differences in the intensity gradients for multiplets of *Fe* I were found by Morgan in several stars of spectral class *A*. The multiplet ($a^3F - y^3F^\circ$) is especially suitable for investigation. Table 2 and Plate I show that the intensity gradient for the three lines λλ 4045, 4063 and 4071 may range from seven units (on an arbitrary scale) for the star 22 Bootis to half a unit for λ Piscium. There appears to be a slight tendency for small gradients to be associated with the more diffuse lines,[4] but this is not true for γ Geminorum and 85 Pegasi, both of which have narrow lines, although their gradients are unusually small. Aside from a slight diffuseness of the lines in several stars, the spectra in Table 2 and in Plate I are perfectly normal for their spectral types.

TABLE 2

(*Fe* I MULTIPLET ($a^3F - y^3F^0$)

	LINE	IQ	LAB.	SPOT	SUN	FLASH	λPSC	δ^1CMI	22BOO	ξSER	d^2VIR	2MON	THEOR.
λ	3969.263	4–3	20III	..	10	2	...	.	...	...	.	...	81
	4005.250	3–2	25 II	6	7	4*N*	4	4	5–6	7	6	5	80
	4045.822	4–4	60rII	28	30	18	4	5	9	7–8	7	8	1215
	4063.604	3–3	45 II	15	20	18	3–4	3	3	4–5	5	4	847
	4071.748	2–2	40 II	15	15	12	3–4	3	2	2–3	3	2	640
	4132.064	3–4	25 II	12	10	7	1	2	2	2	3	2–3	81
	4143.874	2–3	30 I	14	15	6	3–4	4	3–4	4–5	4	4	80

NOTES: All of the stars in the table are of spectral class A5 (H.D.). λ 3969 is masked in A-type stars by *H*ϵ and *H*. λ 4005 may be slightly disturbed in some stars by *V* II 4005.71. λ 4132 may be blended, in stars, with an unknown solar line of intensity 3 at λ 4132.54 and by an *Fe* I line of intensity 4 at 4132.91. λ 4143 is blended, probably not seriously, with *Fe* I 4143.42, in the stars.

4. An effect similar to that previously found for helium has now been discovered for *O* II in the B-type stars.[5] The spectrum of singly-ionized oxygen consists of several systems of doublets and quartets. The latter

predominate in the region usually covered by stellar spectrographs, but there are four lines which are especially convenient for the comparison of their intensities. Table III gives the results for two stars:

TABLE III

LINE	DESIGNATION (FOWLER)	LAB. (FOWLER)	INTENSITY τ SCORPII	INTENSITY ϵ CANIS MAJ.
4345.57	$a^4p_2 - {}^4p_1^1$	7	3	4
4347.43	$a^2d_2^1 - b^2d_2$	5	3	1
4349.44	$a^4p_3 - {}^4p_3^1$	8	6	7
4351.28	$a^2d_3^1 - b^2d_3$	6	5	3

These estimates have been verified on several good spectrograms for each star. While the two quartet lines are slightly stronger in ϵ Canis Majoris, the two doublet lines of this star are much weaker than those of τ Scorpii. ϵ Canis Maj. is a $B1$ giant, while τ Scorpii is a BO dwarf. It is therefore probable that the differences are not caused by temperature; whether they are related to pressure is not certain. The energy levels of these lines do not differ greatly, and it is improbable that the effect is caused by differences in degree of ionization.

5. The intensity gradient within a multiplet depends upon the number of atoms in the lower energy state, the transition probability to the upper state, the number in the upper state, the form of the atomic absorption coefficient, etc. If the absorption coefficient has the form given by Unsöld,

$$\sigma = \frac{\text{Const.}}{(\lambda - \lambda_0)^2}$$

and if the Schuster-Schwarzschild approximation holds,

$$I/I_0 = \frac{1}{1 + \sigma NH,} \tag{1}$$

we find that the total intensity

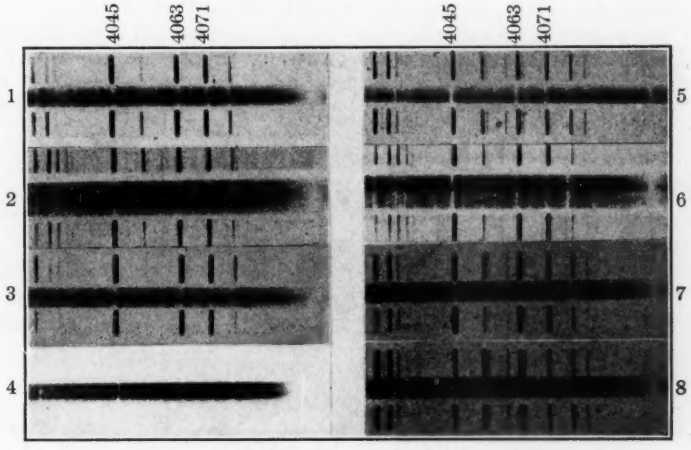
$$E = \int I d\lambda \propto \sqrt{NH}. \tag{2}$$

On the other hand, for lines which are greatly broadened by thermal Doppler effect, Stark effect, etc., the total intensity

$$E \propto NH. \tag{3}$$

W. Schütz[6] has shown that in the laboratory, where radiation damping and thermal Doppler effect are both effective, weak lines show $E \propto NH$, strong lines show $E \propto \sqrt{N}$ and intermediate lines are practically independent of $N.H.$ M. Minnaert and G. F. W. Mulders[7] have found that in the spectrum of the sun lines stronger than Rowland intensity 4 or 5 obey expression (2); lines fainter than Rowland intensity 1 obey expression

(3); and intermediate lines show comparatively little relation between E and $N.H.$ The effective temperature of the sun is approximately 6000°K. We should expect that for hotter stars Doppler broadening would become more effective and would tend to displace the curve of Minnaert and Mulders[8] toward greater intensities. Our observations cannot be explained by this effect: we have found that in stars of the same temperature (or spectral class) the gradient may be large or small. It is possible that causes other than thermal Doppler effect are present; convection currents might produce a similar result and would not necessarily be the same in all stars of similar spectral class. The small gradient in



1. δ^1 Canis Minoris (A5)
2. 72 Ophiuchi (A3)
3. λ Piscium (A5)
4. γ Geminorum (A0)
5. 85 Pegasi (G0)
6. ξ Serpentis (A5)
7. 40 Aurigae (A3)
8. 22 Bootis (A5)

γ Geminorum and the large gradient in ξ Serpentis would not favor this hypothesis, for in both stars the lines are narrow and deep, and their contours resemble the theoretical curves computed on the basis of radiation damping.[9] It does not seem probable, therefore, that the observed differences in gradient are related to the form of the atomic absorption coefficient, neither is it reasonable to postulate that the statistical weights for the terms of a multiplet vary with the conditions.

Much more uncertainty is attached to the number of atoms in various energy levels. The theory of ionization assumes thermodynamic equilibrium, and derives the number of atoms in each level as a function of statistical weight, temperature and pressure. But in deriving equation (1)

we assume that every act of absorption from lower level A to upper level B is followed by an act of emission from B to A. In other words, the number of atoms in state B is defined by the number in A, and by the transition probability $A \longrightarrow B$. This is in contradiction to the assumption of thermodynamic equilibrium. Since in thermodynamic equilibrium the number of atoms in any state is fixed, it seems permissible to interpret our observations as indicating that the reversing layers of stars are not in thermodynamic equilibrium.

[1] O. Struve, *Nature*, **122**, 994 (1928); *Astrophys. J.*, **74**, 248 (1931).

[2] H. N. Russell, *Proc. Nat. Acad. Sci.*, **11**, 314 and 322 (1925).

[3] Struve, *Astrophys. J.*, **76**, 85 (1932).

[4] This tendency, if substantiated, would constitute an important argument against the hypothesis that the broadening of stellar absorption lines is caused *entirely* by axial rotation of the stars. Intensity measurements of a *Si* III triplet in several "n" and "s" stars of class B revealed no differences in the gradient. (Struve and Elvey, *Astrophys. J.*, **72**, 267 (1930).)

[5] See also Struve, *Astrophys. J.*, **74**, 260 (1931).

[6] *Zeits. Astrophysik*, **1**, 300 (1930).

[7] *Ibid.*, **2**, 165 (1931).

[8] Loc. cit., Figure 1.

[9] In order to obtain $E \propto NH$, the lines must be very shallow in the center.

NOTE ON THE DIVISORS OF THE NUMERATORS OF BERNOULLI'S NUMBERS

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We shall consider the Bernoulli numbers as rational fractions in their lowest terms with $B_1 = 1/6$, $B_2 = 1/30$, etc. If l is a prime >3, consider the set

$$B_1, B_2, \ldots, B_{\frac{l-3}{2}}. \tag{1}$$

If l does not divide the numerators of any of these Bernoulli numbers then l is said to be regular, otherwise, irregular. Also if B_i has a numerator divisible by a prime p and p is prime to i, then p is said to be a proper divisor of the numerator; otherwise it is an improper divisor. In this connection we note the von Staudt-Clausen theorem[1] which states that

$$(-1)^n B_n = G_n + \frac{1}{2} + \frac{1}{\alpha} + \frac{1}{\beta} + \ldots + \frac{1}{\lambda}$$

in which $\alpha, \beta \ldots \lambda$ are the numbers obtained by taking the divisors $a, b \ldots l$ of n, setting down the expression $2a + 1, 2b + 1 \ldots 2l + 1$, and selecting only those integers in the latter set which are primes, and G_n is an integer. Hence the denominator of each Bernoulli number is determined. More generally if we set[2]

$$2n = 2^i p^a q^b \ldots r^h s^k \ldots$$

where $p, q \ldots r, s \ldots$ are all primes then the divisors of $2n$ divide into two classes; one class consisting of $2, p, q \ldots$ which appear as divisors in the denominator of B_n and the other class $r^h, s^k \ldots$ appearing in the numerator of B_n as improper divisors.

We also have the congruence[3]

$$\frac{B_n}{n} \equiv (-1)^{k\mu} \frac{B_{n+k\mu}}{n + k\mu} \pmod{l}; \quad \mu = \frac{l-1}{2} \tag{2}$$

for any prime l with n an integer not divisible by $(l - 1)$. We note also that $B_{s\mu}$, where s is a positive integer, has, by the von Staudt-Clausen theorem, l as a divisor of the denominator. Hence, if l is a regular prime, the relation (2) shows that the integer l never appears as a proper divisor of the numerator of any Bernoulli number. On the other hand if l is irregular (2) shows that it is a proper divisor of the numerators of an infinity of Bernoulli numbers.

We have the relation[4]

$$B'_{n+2\mu} - 2B'_{n+\mu} + B'_n \equiv 0 \pmod{l^2}$$

where

$$B'_i = \frac{(-1)^i B_i}{i}.$$

From this we obtain[5]

$$B'_{n+y\mu} \equiv yB'_{n+\mu} - (y-1)B'_n \pmod{l^2}.$$

If we now select l and n so that $B'_n \equiv 0 \pmod{l}$ with $B'_{n+\mu} \not\equiv B'_n \pmod{l}$, it follows that there exists an integer y which yields $B'_{n+y\mu} \equiv 0 \pmod{l^2}$. Pollaczek[6] took $l = 37, n = 16$, which gave $y = 7$; hence $B'_{142} \equiv 0 \pmod{37^2}$. He also found two other examples of this type. Hence the numerator of a Bernoulli number may be divisible by the square of a proper divisor.

Concerning the regular primes, the work of my assistants in recent years has shown that all primes < 307 are regular except

37, 59, 67, 101, 103, 131, 149, 157, 233, 257, 263, 271, 283 and 293. (3)

In this connection Jensen[7] proved that there is an infinity of irregular primes of the form $(4k + 3)$.

We may then state the

THEOREM: *A regular prime never appears as a proper divisor of the numerator of any Bernoulli number. An irregular prime appears as a proper divisor of the numerators of an infinity of Bernoulli numbers. There is an infinity of irregular primes in the form of* $(4k + 3)$. *Contrary to the situation in connection with the divisors of the denominators of the Bernoulli numbers, a numerator of a Bernoulli number may be divisible by the square of a proper divisor. None of the primes* < 307 *ever appear as proper divisors of the numerators of any Bernoulli numbers except those in the set* (3).

We now consider the relation of Bernoulli's numbers to the first factor of the class number of a cyclotomic field $k(\zeta)$, where $\zeta = e^{2i\pi/l}$. This first factor is generally expressed in the form

$$h = \frac{f(Z)f(Z^3)\ldots f(Z^{l-2})}{(2l)^{\frac{1}{2}(l-3)}},$$

$f(x) = r_0 + r_1x + r_2x^2 + \ldots + r_{l-2}x^{l-2}$, $Z = e^{2i\pi/(l-1)}$, r is a primitive root of l and r_i is the least positive residue of r^i, modulo l. The writer has shown[8] from this that

$$h \equiv \frac{l^{\Pi}_{s}(-1)^{(sl^a-1)/2}B_{(sl^a+1)/2}}{2^{\frac{1}{2}(l-3)}} \pmod{l^a} \qquad (4)$$

$s = 1, 3 \ldots l - 2$. This congruence shows that each can be divisible by l^a, with a as large as we please provided certain Bernoulli numbers have numerators divisible by sufficiently high powers of l. On the other hand, *the integer* h *is fixed in value, hence cannot be divisible by an arbitrarily high power of* l. It then follows that we have a limitation on the divisibility of the numerators of certain Bernoulli numbers by powers of l. In particular, it may be shown, as in another paper,[9] that if k of the Bernoulli numbers in the set (1) have numerators divisible by l, then $h \equiv 0 \pmod{l^k}$. Hence, in particular, the value of h possibly limits the value of k. Kummer[10] stated without proof that the asymptotic value of h is

$$\frac{l^{(l-3)/4}}{2^{(l-3)/2}\pi^{(l-1)/2}}.$$

If we should assume Kummer's statement true, this furnishes an obvious limitation on the number of Bernoulli numbers in the set (1) which have numerators divisible by l. Also, the ideas as just expressed, suggest a number of problems. For example, we may use any expression of the type (4) without reference to the class number in order to possibly limit the Bernoulli numbers with numerators divisible by l. It might be feasible to select a much smaller integer than h, which would be the norm of an integer in the field $k(Z)$ and which reduced (mod l^a) to an expression involving Bernoulli numbers.

[1] Bachmann, *Niedere Zahlentheorie*, Vol. II, p. 43.
[2] Bachmann, loc. cit., p. 55.
[3] Bachmann, loc. cit., p. 41.
[4] Bachmann, loc. cit., p. 42.
[5] Pollaczek, *Math. Zeit.*, **21**, p. 28 (1924).
[6] Pollaczek, loc. cit., p. 31.
[7] Jensen, *Nyt Tidsskrift für Matematik* (Afdeling B) 82 (1915).
[8] Vandiver, *Bull. Am. Math. Soc.*, **25**, 458–461 (1919).
[9] Vandiver, *Annals of Math.*, **26** (2), 231–2 (1925).
[10] *J. de Math.*, **16**, 377–488 (1850).

NON-GROUP OPERATIONS

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In order to secure a correct idea of the nature of the group concept it is desirable to consider not only group operations but also some of the closely related non-group operations. One of the most important instances of the latter type of operations from a historical point of view is multiplication by 0. Nicomachus (about 100 A. D.) remarked that 0 added to 0 gives 0, and the Hindus considered the four fundamental operations with respect to 0. If we combine by multiplication the totality of the rational numbers including 0 we do not obtain a group but if we omit 0 from this totality and combine the rest of the set by this operation there obviously results an abelian group of infinite order.

If we say that the distinct elements of a group must obey the associative law and assume that in group theory the operation of combining more than two elements at the same time is not considered, and if we postulate also that when any two of the three symbols in the equation $xy = z$ are replaced by group elements, which need not be distinct, then a third element of the same group is uniquely determined thereby, it results that the rational numbers including 0 obey the associative law but they do not obey the second of the two given group postulates. The following example of three distinct elements which obey the latter of the two given postulates without also obeying the former was given by J. Perott, *American Journal of Mathematics*, volume 11 (1889), page 101.

$1 \cdot 1 = 1$	$1 \cdot 2 = 3$	$1 \cdot 3 = 2$
$2 \cdot 1 = 3$	$2 \cdot 2 = 2$	$2 \cdot 3 = 1$
$3 \cdot 1 = 2$	$3 \cdot 2 = 1$	$3 \cdot 3 = 3$

It may be noted that these three elements are also commutative and

that they obey three of the four group postulates given by H. Weber in the *Mathematische Annalen*, volume 43 (1893), page 522, and later adopted in volume 2 of his *Lehrbuch der Algebra* but not in his *Kleines Lehrbuch der Algebra*, 1912. The three given elements obey also three of the four postulates given in the *Enciclopedia delle Matematiche Elementari*, volume 1, part 2 (1932), page 33. It may be noted that while many writers have started from a set of postulates which either state explicitly or imply that the identity and the inverse of every element of a group must appear therein, S. Lie adopted a more general definition of the term group and was then able to prove that there are groups which involve neither the identity nor the inverse of each of their elements. Cf. S. Lie, *Theorie der Transformationsgruppen*, volume 1 (1888), page 163.

Suppose that when two elements of a group are combined their product is multiplied by the numerical factor $k \neq 0$. It is clear that the given group postulates are satisfied with respect to this operation if each of the elements of a group is supposed to involve also as a numerical factor the various powers of k. In particular, when k is an rth root of unity and the elements of a finite group are multiplied by the various powers of k there results a group which is the direct product of the original group and the cyclic group of order r. On the other hand, if we assume that more than two elements of a group can be combined at the same time and if a product of n of these elements is replaced by the ordinary product of these elements multiplied by n there results a non-group operation. In fact, the second of the two given group postulates is satisfied as regards this operation if each of the elements of the given group is supposed to have for a numerical factor an arbitrary rational number different from 0.

The associative law in group theory is now commonly expressed as follows:

$$a(bc) = (ab)c.$$

Some of the early writers on the subject, including A. Cayley, *American Journal of Mathematics*, volume 1 (1878), page 50, and H. Weber, *Mathematische Annalen*, volume 20 (1882), page 302, stated also that each term of this equation is equal to abc. In the former form the associative law is satisfied by the operation noted near the close of the preceding paragraph but this is obviously not true if the associative law is expressed in the latter form. This exhibits the desirability of emphasizing the fact that in group theory no more than two elements are combined at the same time and hence the expression abc is meaningless therein unless it is assumed that the value thereof is either $a(bc)$ or $(ab)c$. If it is assumed to be equal to one of these two expressions it remains to define the meaning of a product of more than three factors when they are multiplied simultaneously.

The operations of raising a positive number to its different powers and extracting the positive roots of these various powers constitute a group but the operation of root extraction ceased to be a group operation when more than one root of the same number was considered. Hence it results that the operations of raising a given positive number to its various powers and extracting the roots of these powers were regarded as a group until after the time of the ancient Greek civilization, since no instance is known where more than one root was extracted of the same number before the latter part of the early mathematical developments among the Hindus. The more general operations of forming algebraic equations with one unknown and finding a single root of such an equation are not group operations since a root is not always completely determined by such an equation. It has recently been discovered that even the ancient Babylonians sometimes considered more than one root of a given quadratic equation. O. Neugebauer, *Quellen und Studien zur Geschichte der Mathematik-Studien*, volume 2 (1932), page 22.

It is a somewhat singular fact that while the operation of finding a root of a general non-linear algebraic equation is not a group operation the theory of groups arose in connection with this operation. Hence it results that one of the most fruitful algebraic operations is not itself a group operation but group theory has been of fundamental importance in its development and some of its greatest difficulties can be reduced thereto. Even when only positive roots are considered the operation of finding the roots of a general algebraic equation is not a group operation unless the equation is a binomial equation as was noted above. The usefulness of group theory in mathematics cannot be fully determined by a consideration of the group operations which appear therein since non-group operations may depend largely upon this theory.

The operation of constructing geometric figures is also a non-group operation since the construction of two sides of such a figure is not usually equivalent to the construction of a single line segment. The totality of the line segments in a plane constitutes a groupoid but the totality of the vectors therein constitutes a group. While the construction of geometric figures is a non-group operation the movements which transform a regular figure into itself constitute a group and the fact that many of the geometric figures which appear in ancient ornaments admit certain groups of movements has given rise to the view that there may have been a prehistoric group theory, and that the origin of higher mathematics may have appeared 1000 years earlier than has been commonly assumed. Cf. A. Speiser, *Theorie der Gruppen von endlicher Ordnung*, second edition, 1927, page 1. The fact that the first proposition of Euclid's *Elements* relates to a regular figure may be due to the early influence of the group concept.

The operation of finding the limit of an infinite series has been one of the most fruitful operations of all mathematics. While this is not a group operation the theory of continuous transformation groups inaugurated by S. Lie has thrown much new light on this operation. The assumption that only two elements are combined at a time applies to continuous groups as well as to those which are discontinuous. Some operations relating to groups of infinite order have been called group operations by some writers while others have regarded them as non-group operations. An instance of this kind is the totality of the natural numbers when these numbers are combined by multiplication. This is not a group according to the postulates noted above but it has been regarded as a group by many authors including L. Berzolari in the recent encyclopedia noted above, page 43.

ON REPRESENTATION OF INTEGERS BY QUADRATIC FORMS

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This paper is a condensed report on the result of a study of the problem of representation of integers by ternary quadratic forms with integral coefficients. It contains also several theorems on representation of integers by quadratic forms in four or more variables.

We call a quadratic form with integral coefficients classic if and only if the coefficients of the cross products are even, and non-classic if at least one of these coefficients is odd.

I. Indefinite Ternary Classic Quadratic Forms of Quadratfrei Determinant

1. *Statement of Representation Theorems.*—The two theorems stated in this section show that in the case of an indefinite ternary quadratic form f whose determinant is free from square factors, all integers not represented by f form several families of arithmetical progressions depending in a simple fashion on the prime factors of the determinant and the generic characters of f. We distinguish two cases, one of forms with determinant odd and the other of forms with determinant even. Since every indefinite ternary quadratic form with a positive determinant is the negative of a form with a negative determinant, we may state and prove our theorems only for the latter forms.

Theorem 1. *Let f be an indefinite ternary quadratic form of negative*

determinant $-D$ and let F be the reciprocal of f. Let D be a product of distinct positive odd primes. Then the invariants of f are $\Omega = 1$, $\Delta = -D$. Write $D = RE$ in such a way that the characters of f are

$$\begin{aligned}(F\,|\,\rho_i) &= -(-1\,|\,\rho_i)\ (i = 1, \ldots, \alpha),\\ (F\,|\,\pi_j) &= (-1\,|\,\pi_j)\quad (j = 1, \ldots, v)\end{aligned} \tag{1.1}$$

for every prime factor ρ_i of $R = \rho_1 \ldots \rho_\alpha$ and for every prime factor π_j of $E = \pi_1 \ldots \pi_v$, respectively.

If α is even, then f represents every integer a of none of the types

$$\rho_i^{2k+1}[\eta_i\rho_i + \mu(D_i, \rho_i)] \quad (i = 1, \ldots, \alpha) \tag{1.2}$$

and no integers of any of these types. Here $k \geqq 0$, $D_i\rho_i = D$, and

$$\begin{aligned}&\text{for a given integer } x \text{ prime to } p,\\ &\mu(x, p) \text{ runs over all integers}\\ &\text{satisfying } (\mu(x, p)\,|\,p) = (x\,|\,p).\end{aligned} \tag{1.21}$$

If α is odd f represents every integer a of none of the types (1.2) *and*

$$4^k(8n + D) \tag{1.3}$$

where $k \geqq 0$, and no integer of any of these types.

THEOREM 2. *Consider an indefinite ternary quadratic form f of determinant $-D$, where D is double of the product of distinct positive odd primes. The invariants of f are $\Omega = 1$ and $\Delta = -D$. Write F for the reciprocal of f. Let $D = 2RE$, and let the characters of f be*

$$\begin{aligned}(F|\rho_i) &= -(-1|\rho_i)\ (i = 1, \ldots, \alpha);\\ (F|\pi_j) &= (-1|\pi_j)\quad (j = 1, \ldots, v)\end{aligned} \tag{1.4}$$

for every prime factor ρ_i of $R = \rho_1 \ldots \rho_\alpha$ and for every prime factor π_j of $E = \pi_1 \ldots \pi_v$. Then, if α is even, f represents every integer a of none of the types (1.2) *and it represents no integer of any of these types. If α is odd f represents every integer a of none of the types* (1.2) *and*

$$4^k(16n + D) \tag{1.5}$$

where $k \geqq 0$, and no integers of any of these types.

2. *Outline of the Method.*—Our method in outline is as follows: First, using Meyer's[2] criterion for equivalence of indefinite ternary quadratic forms of certain orders, we prove Lemma 1 and employ it to obtain in a very simple fashion the families of arithmetical progressions of integers which are not represented by forms f of a genus K in question. Then we show (in a sequence of three lemmas) that for every integer a not of the types excluded we can construct a form $\Phi = ax^2 + \ldots$ of genus K, having a as the leading coefficient and hence representing a properly. Finally, by Meyer's criterion referred to above, the given form f of the genus K is

equivalent to Φ, therefore f as well as Φ represents a, and the proof is complete.

In case $|\Omega|>1$ the theorems corresponding to Lemma 1 in our case become more complicated and more difficult to handle. The simplicity of the latter, however, together with the sidelights that it furnishes seem to justify the use of it in the present case in preference to a more general method employed by the writer for forms with $|\Omega|>1$.

3. *Lemma 1 and Its Corollaries.*—We shall state Lemma 1 in full.

LEMMA 1. *Consider a positive integer* $D \not\equiv 0 \pmod 4$. *Let* $p_1 \ldots p_n$ *be all the distinct positive odd prime factors of* D. *Then there are* 2^n *classes of properly primitive indefinite ternary quadratic forms with integral coefficients and the invariants* $\Omega = 1$ *and* $\Delta = -D$, *and there exist* 2^n *positive primes* m_i, *each* $\equiv 3 \pmod 4$, *and integers* $e_i = \pm 1$, b_i, c_i *such that the set of* 2^n *forms*

$$f_i = -e_i x^2 + e_i m_i y^2 + 2b_i yz + c_i z^2 \quad (i = 1 \ldots 2^n)$$

contains one and but one representative from each of these classes.

This lemma yields at once the following

COROLLARY. *Every indefinite ternary quadratic form* f *of determinant* D *having the invariants* $\Omega = \pm 1$, *and* $\Delta = D \not\equiv 0 \pmod 4$ *represents at least one of* $+1$ *or* -1, *that is its absolute minimum* $\leqq 1$.

It is of interest to note that Lemma 1 suggests a practical method of constructing the table of representative forms of order $\Omega = \pm 1$, $\Delta = D$ through determination of $e_i m_i$. Here m_i need not be a prime. The method is especially useful in case D is very large and Markoff's[3] method cannot be applied. For example, let $D = pq$, where p and q are positive odd primes each $\equiv 3 \pmod 4$ and such that $p \not\equiv q \pmod 8$. Then every indefinite ternary quadratic form of determinant D (however large D may be) is equivalent to one of the four forms

$$f_1 = x^2 + y^2 - Dz^2, \quad f_2 = -x^2 + 2y^2 + 2yz - \frac{D-1}{2} z^2,$$

$$f_3 = -x^2 - y^2 + Dz^2, \quad f_4 = -x^2 - 2y^2 + 2yz + \frac{D-1}{2} z^2,$$

and no two of these forms are equivalent. Form f_1 represents zero properly. Forms f_2, f_3, f_4 have absolute minimum equal to unity.

II. GENERAL THEORY FOR TERNARY QUADRATIC FORMS

4. We establish a very simple relation between the generic characters of a ternary quadratic form f and the arithmetical progressions of integers (not necessarily prime to the determinant D of f) not represented by f. The remaining integers are shown to be represented by some form[4] in the

genus to which f belongs. In the case of indefinite forms we employ Meyer's[5] criterion for equivalence to obtain general theorems on representation.

5. *Integers Not Represented by a Form of a Given Genus.*—We consider properly or improperly primitive ternary quadratic forms

$$f = ax^2 + by^2 + cz^2 + 2ryz + 2sxz + 2txy \qquad (5.11)$$

where $a \ldots t$ are integers. Let Ω, Δ be the invariants and

$$F = Ax^2 + By^2 + Cz^2 + 2Ryz + 2Sxz + 2Txy \qquad (5.12)$$

be the reciprocal of f. Multiplying both members of (5.11) by a we obtain

$$af = X^2 + \Omega Cy^2 - 2\Omega Ryz + \Omega Bz^2, \; X = ax + ty + sz. \quad (5.13)$$

We may assume $(a, \Omega) = 1$ or 2. Then (5.13) shows that for every odd prime factor ω of Ω, the value of Legendre's symbol $(m \mid \omega)$ is the same for every integer m represented by f and prime to ω, viz., $(m \mid \omega) = (a \mid \omega)$. Similarly if 4 or 8 divides Ω, $m \equiv a$ holds (mod 4) or (mod 8), respectively, for every odd m. A reciprocity relation between f and F permits us to conclude at once a similar result for integers M represented by F relative to prime factors δ or 4 or 8 of Δ. Relation (5.13) may thus serve as a means for a simple introduction of generic characters of ternary forms.

We shall use classic notations $(f \mid \omega)$, $(F \mid \delta)$, $(-1)^{\frac{f-1}{2}}$, $(-1)^{\frac{F-1}{2}}$, $(-1)^{\frac{f^2-1}{8}}$, $(-1)^{\frac{F^2-1}{8}}$ for the generic characters of f. We have just shown that f does not represent an integer m prime to ω such that

$$(m \mid \omega) = -(f \mid \omega).$$

It remains to consider integers divisible by ω. Write

$$m = m_1\omega^k, \; \Omega = \Omega_1\omega^l$$

where m_1 and Ω_1 are prime to ω. Restrictions on m_1 and k are obtained very simply from (5.13). For example let $l > k$. Since (5.13) yields

$$am_1\omega^{l} \equiv [illegible] \pmod{\omega^l},$$

m is not represented if k is odd or it κ is even and $(m_1|\omega) = -(f|\omega)$.

Next multiply (5.13) by C. We get

$$Caf = CX^2 + \Omega Y^2 + \Omega\Delta az^2 \qquad (5.21)$$

where

$$X = ax + ty + sz, \; Y = Cy - Rz. \qquad (5.22)$$

Equation (5.21) permits us to find very simply progressions depending on certain prime factors δ of Δ, of integers not represented by f. We may

assume without loss of generality that a and C are relatively prime and have no odd common factor with $\Omega\Delta$. Then

$$(F \mid \delta) = (C \mid \delta) \tag{5.23}$$

for every odd prime factor δ of Δ. Also, in case 4 or 8 divides Δ

$$(-1)^{\frac{F-1}{2}} = (-1)^{\frac{C-1}{2}}, \quad (-1)^{\frac{F^2-1}{8}} = (-1)^{\frac{C^2-1}{8}}.$$

We shall consider δ not dividing Ω (all others having been taken care of by (5.13)) and such that

$$(F \mid \delta) = -(-\Omega \mid \delta). \tag{5.24}$$

Let $m = \delta m_1$. Then the equation (5.21) with f replaced by m yields

$$CX^2 + \Omega Y^2 \equiv Cam_1\delta \equiv 0 \pmod{\delta}.$$

Hence, by (5.24), $X \equiv Y \equiv 0 \pmod{\delta}$. Write $X = \delta X_1$, $Y = \delta Y_1$. Substituting into (5.21) and dividing through by the common factor δ we obtain

$$Cam_1 \equiv \Omega \frac{\Delta}{\delta} az^2 \text{ or } \Omega\Delta az^2/\delta \pmod{\delta}.$$

Since a is prime to δ and by (5.23) and (5.24), m is not represented if δ^2 does not divide Δ and $(m_1 \mid \delta) = (-\Delta/\delta \mid \delta)$, or if δ^2 divides Δ and m_1 is prime to δ. Similar arguments yield restrictions on $k > 1$ and m_1 when $m = \delta^k m_1$. We note that if m is not represented by f, and δ^2 does not divide Δ, then $\delta^2 m$ is not represented by f. For otherwise $X \equiv Y \equiv z \equiv 0 \pmod{\delta}$ in (5.21) with f replaced by $m\delta^2$ whence by (5.22) $x \equiv y \equiv z \equiv 0 \pmod{\delta}$ and f would represent m.

It remains now to consider exclusion modulo a power of 2. We have already seen that divisibility of Ω by 4 or 8 imposes restrictions modulo 4 or 8 on odd integers represented by f. Further study of (5.21) modulii 8 and 16 yields conditions analogous to (1.3) and (1.5).

6. *Universal Forms.*—A positive form representing all positive integers and an indefinite form representing all positive or negative integers are said to be universal. Methods of the preceding section show that no positive ternary quadratic form is universal. These methods also give necessary conditions for universality of indefinite ternary quadratic forms. In this section we state conditions which are not only necessary but also sufficient. If f is a properly primitive classic form, conditions for its universality are expressed in terms of its generic characters. In the case of a primitive non-classic form Ψ, we study the improperly primitive classic form $f = 2\Psi$ and deduce the desired conditions in terms of the generic characters of f.

THEOREM 3. *Let f be a properly primitive indefinite classic ternary quadratic form with reciprocal F and determinant D. Necessary and sufficient conditions that f be universal are*

$$D \text{ is odd or double and odd integer, } \Omega = \pm 1 \tag{6.25}$$

and

$$(F \mid p) = (-\Omega \mid p) \tag{6.26}$$

for every odd prime p dividing $\Delta = D$.

THEOREM 4. *Let f_1 be a primitive indefinite non-classic ternary quadratic form. Consider an improperly primitive form $f = 2f_1$. Let F be the reciprocal and Ω, Δ the invariants of f. Then f_1 is universal if and only if $f = 2f_1$ satisfies the following conditions:*

$$\Omega = \pm 1, \tag{6.27}$$

the characters of f are

$$(-1)^{\frac{F-1}{2}} = (-1)^{\frac{-\Omega-1}{2}}, \quad (F \mid p) = (-\Omega \mid p) \tag{6.28}$$

for every odd prime p dividing $\Delta = D$, and, in case 4 divides Δ,

$$(-1)^{\frac{F^2-1}{2}} = 1. \tag{6.29}$$

It is of interest to note that the above theorems serve as an immediate practical test for determining whether a given ternary quadratic form is or is not universal.

The proof that conditions (6.25) – (6.29) are sufficient runs as follows. We reduce the problem to that of representation of multiples of an appropriately chosen integer a equal to a prime or double a prime, by a form of the type $x^2 - B(y, z)$ where $B(y, z)$ is a primitive binary form of determinant aD and of the principal genus. Then (after Oppenheim[6]) we show that such ternary form is equivalent to one which is obviously universal.

Dickson[7] noted that

THEOREM 5. *Every universal, classic or non-classic, ternary quadratic form is a zero form.*

We prove this theorem by observing that since f or $\frac{1}{2}f$ is universal, a in (5.21) can be taken equal to $-\Delta_1$ or $-4\Delta_1$ where Δ_1 is the quotient of Δ by the largest square dividing it (both $-\Delta_1$ and, in case f is improperly primitive, $-4\Delta_1$ are represented properly).

Every zero form is equivalent to (5.11) with $a = s = 0$. The determinant of such a form is $-ct^2$. If the determinant of f is free from square factors, then $t = 1$ and

$$f \sim by^2 - Dz^2 + 2ryz + 2xy.$$

Replace x by $x + ky - rz$. Then, by choice of k, f is equivalent to either

$$2xy + y^2 - Dz^2 \text{ or } 2xy - Dz^2,$$

and hence f is universal. This proves

THEOREM 6. *Every zero ternary quadratic form of quadratfrei determinant is universal.*

7. *Integers Represented by Some Form in the Genus.*—Having shown that certain arithmetical progressions of integers are not represented by a form f of a genus K in question we next show that every other integer is represented by at least one form in this genus. To do this latter we have an alternative of two methods. We either show that for every integer a not of the types excluded we can construct a form $\Phi = ax^2 + \ldots$ of genus K, having a as the leading coefficient and hence representing a properly. Or we can construct a form f such that the desired integer is the third coefficient of the reciprocal F of f, and F belongs to K.

A. Meyer[8] stated conditions under which there is but one class in a genus of certain orders Ω, Δ. Application of these conditions to the results obtained by methods of this section and section five yields general theorems on representation analogous to Theorems 1 and 2, but involving more complicated conditions modulii ω_i, δ_i, and powers of 2.

8. *Reduction to Sum of Squares.*—The following theorem seems to be of interest inasmuch as every primitive form (5.11) is equivalent to one with a and C equal to distinct primes or doubles of primes not dividing the discriminant of f.

THEOREM 7. *Employ notations of section* 5. *Let each of a and C be a prime or a double of a prime not dividing* $\Omega\Delta$. *Consider a form*

$$G = CX^2 + \Omega Y^2 + \Omega\Delta aZ^2. \tag{8.11}$$

If f represents an integer m, then G obviously represents the multiple Cam of m. Conversely *if Cam is represented by G then m is represented by f.*

This theorem has been generalized to apply to quadratic forms in n variables.

The proof of Theorem 7 makes use of the fact that if a difference of the squares is congruent to zero modulo a prime (or a double of a prime), then either sum or the difference of the bases is divisible by this prime (or its double). The squares in question are Y^2 and $R^2Z^2 \equiv -\Delta aZ^2 \pmod{C}$ in (8.11) and X^2 and $(ty + sz)^2 \equiv -\Omega Cy^2 - 2\Omega Ryz - \Omega Bz^2 \pmod{a}$ in (5.13).

Theorem 7 reduces the problem of representation of integers by a general positive or indefinite ternary quadratic form (5.11) of invariants Ω, Δ to that of representations of multiples of appropriately chosen integers a and C by (8.11) containing no cross products. In the case of positive forms this theorem permits a similar reduction of the problem of determination of the number of representations.

III. QUATERNARY QUADRATIC FORMS

9. *Positive Universal Quaternary Forms.*—It is a classic fact that the form $x^2 + y^2 + z^2 + u^2$ represents all positive integers. Examples of other forms

$$\Phi = ax^2 + by^2 + cz^2 + du^2 \tag{9.11}$$

with positive integral coefficients a, b, c, d which represent all positive integers were first obtained by Jacobi,[9] Liouville[10] and Pepin.[11] Ramanujan[12] proved that there are only 55 sets of positive integers a, b, c, d such that (9.11) represents all positive integers. The present writer considered a general positive quaternary form $Q(x, y, z, u)$ and showed that there is but a finite number of positive universal forms (counting but one representative from each class). In fact

THEOREM 8. *There exists no universal positive quaternary form of determinant* > 112.

The above serves as an indirect proof of the statement made in Section 6, that no positive ternary quadratic form represents all positive integers.

When we turn from positive to indefinite forms we find an entirely different situation.

10. *Certain Indefinite Quaternary Forms.*—C. G. Latimer[13] remarked that representation of integers by forms (9.11) with coefficients a, b, c, d not all of the same sign had not been studied, and proceeded to study the form

$$Q = x^2 + y^2 - \alpha z^2 - \alpha\omega^2, \quad \alpha > 0. \tag{10.11}$$

He showed that when $\alpha < 163$, is free from square factors, and is of the form $4k + 3$, Q is universal. He also studied even α's for the same range. R. H. Marquis[14] extended the above result to $\alpha < 81$ of the form $4k + 1$ and also for some even α's.

Employing Theorem 1, we prove

THEOREM 9. *The form* (10.11) *is universal for every* α *which is a product of distinct odd primes.*

This theorem can be generalized. We shall, however, employ theorems of Section 1 to prove a more useful

THEOREM 10. *Consider quaternary form*

$$Q = \omega^2 + \Phi(x, y, z) \tag{10.12}$$

where $\Phi(x, y, z)$ *is a primitive indefinite ternary form of quadratfrei determinant. Such a form* Q *represents all positive and all negative integers* m.

The proof of the last two theorems consists in showing that an integer ω can be chosen so that $\alpha\omega^2 + m$ and $m - \omega^2$ in the respective theorems are of none of the types excluded in Theorems 1 and 2.

11. *Forms of Quadratfrei Determinant.*—Let

$$f_{(n)} = \sum_{ij}^{1,n} a_{ij}x_ix_j \tag{11.11}$$

of determinant D be a primitive quadratic form in n variables. Multiplying (11.11) by a_{11} we obtain

$$a_{11}f_{(n)} = \left(\sum_{j}^{1,n} a_{1j}x_j\right)^2 + f_{(n-1)}(x_2, \ldots, x_n) \tag{11.12}$$

where the coefficients of $f_{(n-1)}$ are the two rowed minors of D with a_{11} as their upper left-hand corner element.

THEOREM 11. *Consider form*

$$G_{(n)} = X^2 + f_{(n-1)}(x_2, \ldots, x_n). \tag{11.13}$$

Let a_{11} be a prime or double a prime. Then if the form $f_{(n)}$ represents m, $G_{(n)}$ obviously represents $a_{11}m$, and conversely *if $G_{(n)}$ represents $a_{11}m$ then $f_{(n)}$ represents m.*

The proof of this is a simple generalization of the second part of the proof of Theorem 7.

Let $f_{(4)}$ be an indefinite quaternary form. We may assume that a_{11} is a prime or double a prime and prime to D. By Theorem 11, $f_{(4)}$ is universal *if and only if* $G_{(4)}$ represents all multiples of a_{11}. If determinant D of $f_{(4)}$ is free from square factors and is positive then $f_{(3)}$ in (11.13) is an indefinite ternary quadratic form of quadratfrei determinant. In view of Theorem 10 this proves

THEOREM 12. *Every indefinite properly primitive quaternary form of positive quadratfrei determinant is universal.*

Theorem 12 is an instance of a more general theorem which is to appear elsewhere.

[1] NATIONAL RESEARCH FELLOW.

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[9] Jacobi (see *Dickson's History*, **2**, p. 263).

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